5. M. Jakob and W. Linke, Der Wärmeübergang beim Verdampfen von Flüssigkeiten an senkrechten und waagerechten Flächen. *Phys. Z.* **36**, 267–280 (1935).

6. K. Yamagata, F. Hirano, K. Nishikawa and H. Matsuoka, Nucleate boiling of water on the horizontal heating surface. *Mem. Fac. Engng. Kyushu*, **15**, 97–163 (1955).

TURBULENT HEAT TRANSFER IN A TUBE WITH CIRCUMFERENTIALLY-VARYING TEMPERATURE OR HEAT FLUX

E. M. SPARROW and S. H. LIN

University of Minnesota, Minneapolis, Minnesota

(*Received* 21 *February* 1963)

Although both experiment and analysis of turbulent heat transfer in a circular tube are generally carried out under the assumption that temperature and heat flux are circumferentially uniform, there are many practical situations where the thermal boundary conditions vary around the tube circumference. For instance, such variations may arise if the tube is externally heated (or cooled) by a forced-convection crossflow, by free convection, by film condensation, or by thermal radiation from a non-uniform environment. This paper presents a brief account of an analysis of turbulent heat transfer with circumferentially-varying temperature or heat flux. It is expected that experiments will be made later to check on the nalysis.assumptions of the a

Consider a hydrodynamically and thermally fully-developed tube flow in which there is a uniform heat-transfer rate per unit length Q, but in which the wall temperature T_w may vary around the circumference in an arbitrary manner as given by a Fourier series

$$\frac{[T_w(\varphi) - \bar{T}_w]}{(\bar{q}r_o/k)} = \sum_{n=1}^{\infty} a_n \cos n\varphi + b_n \sin n\varphi \tag{1}$$

in which φ is the angular co-ordinate, r_o the tube radius, $\bar{q}$ the average heat flux rate per unit area ($\bar{q} = Q/2\pi r_o$), and $\bar{T}_w$ the circumferentially-averaged wall temperature at some axial position. It will be shown later that the circumferential variation of the heat flux could equally well be prescribed in lieu of $T_w(\varphi)$.

The analysis begins with the energy equation for fully-developed flow and heat transfer ($\partial T/\partial z = -2\bar{q}/r_o\bar{u}\rho c_p$, $\bar{q}$ is positive from fluid to wall)

$$-\left(\frac{4}{r_o{}^+ RePr}\right) r^+u^+ = \frac{\partial}{\partial r^+}\left[r^+\left(\frac{1}{Pr} + \frac{\epsilon_{h,\,r}}{\nu}\right)\frac{\partial\theta}{\partial r^+}\right] + \frac{1}{r^+}\frac{\partial}{\partial\varphi}\left[\left(\frac{1}{Pr} + \frac{\epsilon_{h,\,\varphi}}{\nu}\right)\frac{\mathrm{d}\theta}{\partial\varphi}\right] \tag{2}$$

in which

$$\theta = \frac{(T - \bar{T}_w)}{(\bar{q}r_o/k)}, \quad r^+ = \frac{r\sqrt{(\tau_w/\rho)}}{\nu}, \quad u^+ = \frac{u}{\sqrt{(\tau_w/\rho)}} \tag{3}$$

and Re and Pr are the Reynolds and Prandtl numbers, τ_w is the wall shear. It is the belief of the authors that the turbulent transport is set up by the action of the flow field, which is axisymmetric. Consistent with this, the eddy diffusivities for heat $\epsilon_{h,\,r}$ and $\epsilon_{h,\,\varphi}$ will be taken to be independent of φ.

Although information on the radial eddy diffusivity $\epsilon_{h,\,r}$ is available, there appears to be no information on the circumferential diffusivity, $\epsilon_{h,\,\varphi}$. As an initial assumption, $\epsilon_{h,\,r} = \epsilon_{h,\,\varphi}$ is not unreasonable.* Expressions for the eddy diffusivity and the velocity distribution, previously employed in an analysis of axisymmetric heat transfer [1] which has been checked by experiments for $Pr = 0{\cdot}7$ and $Pr = 7{\cdot}0$, will also be used here.

The solution of equation (2) for θ can be written as

$$\theta(r^+, \varphi) = \theta_{fd}(r^+) + \sum_{n=1}^{\infty} R_n(r^+)\,(a_n \cos n\varphi + b_n \sin n\varphi) \tag{4}$$

in which θ_{fd} is the fully-developed temperature solution for axisymmetric heating and the R_n functions depend only on r^+. The R_n are found by solving

$$\frac{\mathrm{d}}{\mathrm{d}r^+}\left[r^+\left(\frac{1}{Pr} + \frac{\epsilon}{\nu}\right)\frac{\mathrm{d}R_n}{\mathrm{d}r^+}\right] - \frac{n^2}{r^+}\left(\frac{1}{Pr} + \frac{\epsilon}{\nu}\right) R_n = 0,$$
$$R_n(0) = 0,\ R_n(r_o{}^+) = 1. \tag{5}$$

This two point boundary value problem has been solved by applying a numerical integration over most of the cross section and matching this with an analytic series solution near $r^+ = 0$ (where a numerical solution fails). Representative results will be shown later.

The local heat flux may be obtained by differentiating (4)

$$\frac{q}{\bar{q}} = 1 - r_o{}^+ \sum_{n=1}^{\infty}\left(\frac{\mathrm{d}R_n}{\mathrm{d}r^+}\right)_{r_o{}^+} (a_n \cos n\varphi + b_n \sin n\varphi). \tag{6}$$

The variation of q with φ can thus be calculated. Conversely, if $q/\bar{q}$ is prescribed and $\mathrm{d}R_n/\mathrm{d}r^+$ is available from the solutions of equation (5), then the circumferential wall temperature variation may be found by applying

* Personal communication, Professor G. K. Batchelor, Cambridge University, June, 1962.

equations (6) and (1). A circumferentially-averaged Nusselt number $\overline{Nu}$ may be defined and evaluated as

$$\overline{Nu} = \left[\frac{\int_0^{2\pi} q\,\mathrm{d}\varphi}{2\pi\,(T_b - \bar{T}_w)}\right]\frac{2r_o}{k} = Nu_{fd}. \tag{7}$$

In other words, the average heat-transfer performance is equal to that of the axisymmetric heat-transfer situation.

Illustrative calculations have been carried out for $Pr = 0{\cdot}7$ for $Re \simeq 50\,000$ ($r_o^+ = 1290$) and $Re = 150\,000$ ($r_o^+ = 3380$). The R_n functions were determined for $n = 1$ through $n = 6$ and these are shown in Fig. 1. The

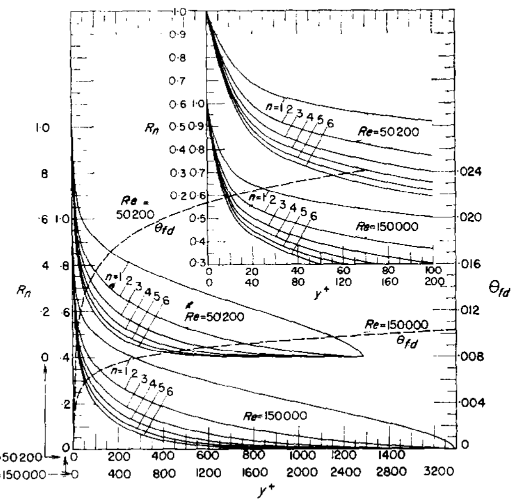

FIG. 1. Distributions of R_n and θ_{fd} for $Pr = 0{\cdot}7$, $Re = 50\,200$ and $150\,000$.

y^+ co-ordinate is equal to $(r_o^+ - r^+)$. An inset gives results for the range of small y^+. The figure indicates that the R_1 function affects the temperature profile over the entire cross section, while the higher R_n are important only near the wall. The axisymmetric, fully-developed temperature profiles θ_{fd} are shown as dashed curves. Also, the derivatives $(\mathrm{d}R_n/\mathrm{d}r^+)_{r_o^+}$ needed for the heat-transfer calculation are listed in Table 1.

Table 1. *Values of* $(\mathrm{d}R_n/\mathrm{d}r^+)_{r_o^+}$, $Pr = 0{\cdot}7$

$n \rightarrow$	1	2	3	4	5	6
$Re = 50\,200$	2·392	3·127	3·496	3·749	3·946	4·109
$Re = 150\,000$	2·185	2·788	3·079	3·274	3·424	3·547

Fig. 2 shows results for an illustrative case where in $(T_w - \bar{T}_w)/(\bar{q}r_o/k) = 0{\cdot}1 \cos\varphi$, i.e. the temperature variation around the tube is about equal to the wall-to-bulk temperature difference. Temperature profiles are plotted as a function of angular position φ at various distances from the wall. The circumferential variations are damped out in the core of the flow. However, due to the assymmetry, the centerline temperature is not the highest in the cross section. The wall heat flux varies about ± 30 per cent around the mean.

Interested readers are referred to the laminar flow study by Reynolds [2].

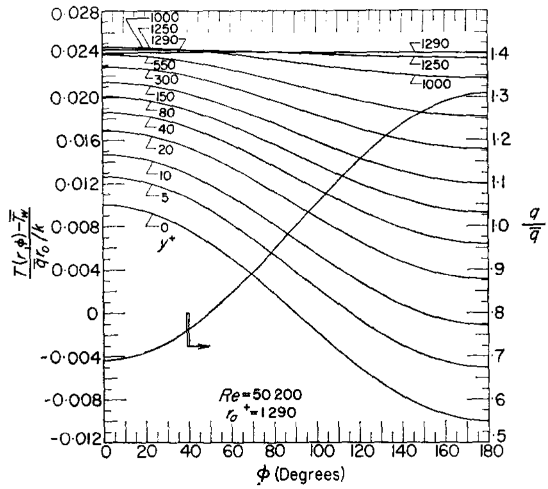

FIG. 2. Illustrative circumferential variations of fluid temperature and of wall heat flux corresponding to a prescribed wall temperature variation, $Pr = 0{\cdot}7$, $Re = 50\,200$.

ACKNOWLEDGEMENT

The support of the U.S. Atomic Energy Commission under contract AT (11–1)-659 is gratefully acknowledged.

REFERENCES

1. E. M. SPARROW, T. M. HALLMAN and R. SIEGEL, Turbulent heat transfer in the thermal entrance region of a pipe with uniform heat flux, *Appl. Sci. Res.* A **7**, 37–52 (1957).
2. W. C. REYNOLDS, Heat transfer to fully-developed laminar flow in a circular tube with arbitrary circumferential heat flux, *J. Heat Transfer*, C **82**, 108–112 (1960).